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Short Communication

Superaccurate finite element eigenvalues via a Rayleigh quotient correction

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Abstract

The consistent finite element formulation of the vibration problem generates upper bounds on the corresponding exact eigenvalues but requires the solution of the highly expensive general algebraic eigenproblem $Kx = \lambda Mx$ with a global matrix M that is of the same sparsity pattern as the global stiffness K. The lumped, diagonal, mass matrix finite element formulation is no longer variationally correct but results in a simplified algebraic eigenproblem of comparable accuracy. We may write the mass matrix as a linear matrix function, $M(\gamma) = M_1 + \gamma M_2$, of parameter γ such that $M(\gamma = 1)$ is the (diagonal) lumped mass matrix and $M(\gamma = 0)$ is the consistent mass matrix. It has been shown that an optimal γ exists between these two states which results in superaccurate eigenvalues. What detracts from the appeal of this approach is that the superior accuracy thus achieved comes at the hefty price of having to solve the still general algebraic eigenproblem with a nondiagonal mass matrix. In this note we show that the same superior accuracy can be had by first computing an eigenvector u from $Ku = \lambda Du$, in which $D = M_1 + M_2$ is the lumped, diagonal, mass matrix, and then obtaining the corresponding, superaccurate, eigenvalue from the Rayleigh quotient $R[u] = u^{\mathrm{T}}Ku/u^{\mathrm{T}}M(\gamma)u$, $M(\gamma) = M_1 + \gamma M_2$ for an optimal γ.

1. Introduction

In a recent note [1] we suggested (see also Refs. [2,3]) to write the finite element mass matrix as a linear function, $M(\gamma) = M_1 + \gamma M_2$, of parameter γ, such that $M(\gamma = 1)$ is the (diagonal) lumped

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mass matrix and $M(\gamma=0)$ is the consistent mass matrix, and adjust γ to achieve super-convergence. What detracts from the appeal of this approach is that the superior accuracy thus achieved comes at the hefty price of having to solve the general algebraic eigenproblem $Ku=\lambda Mu$ with a mass matrix M that is not diagonal. The purpose of this note is to show that the same superior accuracy can be had by first computing an eigenvector u from $Ku=\lambda Du$, in which $D=M_1+M_2$ is the lumped *diagonal* matrix, and then obtaining the corresponding, super-accurate, eigenvalue from the Rayleigh quotient

$$R[u]=\frac{u^{\mathrm{T}}Ku}{u^{\mathrm{T}}M(\gamma)u},\quad M(\gamma)=M_1+\gamma M_2 \tag{1}$$

for an optimal γ.

2. Two-nodes string element

As before we start with looking at the simplest, most accessible to analysis, model problem of the vibrating unit string, described by the boundary value problem

$$u''+\lambda u=0,\quad 0<x<1,\quad u(0)=u(1)=0 \tag{2}$$

for which $\lambda=\pi^2$ is the lowest eigenvalue, and $u=u(x)=\sin\pi x$ the corresponding eigenfunction.

The linear element matrices for the string problem are

$$k=\frac{1}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix},\quad m(\gamma)=\frac{h}{6}\left(\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}+\gamma\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right) \tag{3}$$

with k being the element stiffness matrix, and with $m(\gamma)$ being the element mass matrix such that

$$m(0)=\frac{h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\quad\text{and}\quad m(1)=\frac{h}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{4}$$

are the consistent and the lumped element mass matrices, respectively.

Let the interval $0\leqslant x\leqslant 1$ be divided into $n+1$ sections of size $h=1/(n+1)$ with nodes labeled $0,1,2,\ldots,n,n+1$. Assembly of the linear finite elements over this mesh leads to the typical finite difference equation

$$u_j-2u_{j+1}+u_{j+2}+\frac{\lambda h^2}{6}((1-\gamma)u_j+(4+2\gamma)u_{j+1}+(1-\gamma)u_{j+2})=0,\quad j=0,1,\ldots,n,n+1. \tag{5}$$

The characteristic equation of finite differences scheme (5) is

$$z^2+2z\frac{-1+\frac{1}{6}(2+\gamma)\lambda h^2}{1+\frac{1}{6}(1-\gamma)\lambda h^2}+1=0, \tag{6}$$

so that for the fundamental mode and the lowest eigenvalue

$$\cos\pi h=\frac{2-\frac{1}{6}(4+2\gamma)\lambda h^2}{2+\frac{1}{6}(2-2\gamma)\lambda h^2} \tag{7}$$

and consequently

$$\lambda = \frac{6}{1-\gamma}\frac{1}{h^2}\frac{1-\cos \pi h}{\beta+\cos \pi h}, \quad \beta = \frac{2+\gamma}{1-\gamma}, \quad \gamma \neq 1. \tag{8}$$

Power series expansion produces

$$\lambda = \pi^2[1+(1-2\gamma)\pi^2 h^2] + O(h^4) \tag{9}$$

and the choice $\gamma = \frac{1}{2}$ results in

$$\lambda = \pi^2\left(1 - \frac{\pi^4 h^4}{240}\right) + O(h^6) \tag{10}$$

with a superior asymptotic accuracy $O(h^4)$.

3. A Rayleigh quotient correction

For the lumped matrix formulation ($\gamma = 1$) we obtain from Eq. (5) and the end conditions $u(0) = u(1) = 0$ the first eigenvector

$$u_j = \sin\left(\pi h \frac{j}{n+1}\right), \quad j = 0, 1, \ldots, n, n+1 \tag{11}$$

which we observe to be the interpolant to the first eigenfunction $u(x) = \sin \pi x$, $0 \leqslant x \leqslant 1$. From Eq. (7) we obtain, with $\gamma = 1$, the corresponding lowest eigenvalue

$$\lambda = \frac{1}{h^2}(1-\cos \pi h) = \pi^2(1 - \tfrac{1}{12}x^2 + \tfrac{1}{360}x^4 + \cdots), \quad x = \pi h. \tag{12}$$

We now propose to use eigenvector u of Eq. (11) to compute a new approximation to λ using the Rayleigh quotient

$$R[u] = \frac{u^{\mathrm{T}}Ku}{u^{\mathrm{T}}M(\gamma)u} = \frac{u^{\mathrm{T}}Ku}{u^{\mathrm{T}}M_1 u + \gamma u^{\mathrm{T}}M_2 u}. \tag{13}$$

At this point we prefer to make the substitution $\gamma = 1 - \delta$ so as to have

$$R[u] = \frac{u^{\mathrm{T}}Ku}{u^{\mathrm{T}}M_L u - \delta u^{\mathrm{T}}M_K u}, \tag{14}$$

where M_L is the lumped mass matrix, and

$$M_K = \frac{h^2}{6}K, \tag{15}$$

K being the global stiffness matrix. Hence

$$R[u] = \frac{u^{\mathrm{T}}Ku/u^{\mathrm{T}}M_L u}{1-(\delta h^2/6)(u^{\mathrm{T}}Ku/u^{\mathrm{T}}M_L u)} = \frac{\lambda}{1-(\delta h^2/6)\lambda}, \tag{16}$$

where $\lambda = u^{\mathrm{T}}Ku/u^{\mathrm{T}}M_L u$ is the lowest eigenvalue computed with the lumped mass matrix M_L.

Substituting λ from Eq. (12) into Eq. (14) we obtain

$$R[u] = \pi^2 \frac{1 - \frac{1}{12}x^2 + \frac{1}{360}x^4 - \cdots}{1 - (\delta/6)(x^2 - \frac{1}{12}x^4 + \cdots)}, \quad x = \pi h \tag{17}$$

indicating that for highest accuracy $\delta = \gamma = \frac{1}{2}$. For this choice of γ

$$R[u] = \pi^2 \frac{1 - \frac{1}{12}x^2 + \frac{1}{360}x^4 - \cdots}{1 - \frac{1}{12}x^2 + \frac{1}{144}x^4 + \cdots} = \pi^2(1 - \tfrac{1}{240}x^4) + O(x^6), \quad x = \pi h \tag{18}$$

which is exactly the result we directly obtain for $\gamma = \frac{1}{2}$.

If generally true, then this conclusion is of considerable computational interest. It means that the $O(h^4)$ superaccuracy in the computed λ can be achieved by first computing the eigenvectors for the lumped finite element version of the mass matrix, then computing eigenvalue λ from Rayleigh's quotient using the optimal mass matrix. Solving $Ku = \lambda u$ requires a sophisticated iterative procedure, while the computation of Rayleigh's quotient does not require more that matrix vector multiplications that can be performed even without the explicit assembly of the finite elements global matrices.

4. Quadratic string element

The element stiffness and mass matrices for a three-nodes quadratic string element of size $2h$ are

$$k = \frac{1}{6h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \quad \text{and} \quad m(\gamma) = \frac{h}{15}\left(\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \gamma \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}\right), \tag{19}$$

respectively. We write $m(\gamma) = m_1 + \gamma m_2$, and obtain

$$m(0) = \frac{h}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad m(1) = \frac{h}{3}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 1 \end{bmatrix} \tag{20}$$

as the consistent and lumped element matrices, respectively.

We observe that

$$u^{\mathrm{T}}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} u = (u_1 - 2u_2 + u_3)^2 \tag{21}$$

for any vector $u = [u_1 u_2 u_3]^{\mathrm{T}}$ and we recall the fact that $(u_1 - 2u_2 + u_3)/h^2$ is a finite difference formula for the *second* derivative of the function $u = u(x)$. In particular, if $u_j = \sin(j\pi h)$, then

$$(u_0 - 2u_1 + u_2)/h^2 = \frac{1}{h^2}(0 - 2\sin \pi h + \sin 2\pi h) = -\pi^3 h \tag{22}$$

if $h \ll 1$.

In Ref. [1] it has been shown that for the optimal $\gamma = \frac{2}{3}$ the accuracy of the computed λ becomes $O(h^6)$ instead of the $O(h^4)$ accuracy of the consistent and lumped finite element matrix formulations.

For a fixed vector Rayleigh's quotient is a rational function of γ and we propose to explore now the highest accuracy that can be obtained from this, γ dependent, quotient for eigenvector u computed with the lumped finite element mass matrix formulation.We make the substitution $\gamma = 1 - \delta$ and have

$$R[u] = \frac{u^{\mathrm{T}} K u}{u^{\mathrm{T}} M_L u - \delta u^{\mathrm{T}} M_K u}, \tag{23}$$

where M_L is the lumped mass matrix, and M_K is the global matrix assembled from the m_2 part of the element mass matrix in Eq. (19).

We rewrite $R[u]$ as

$$R[u] = \frac{u^{\mathrm{T}} K u / u^{\mathrm{T}} M_L u}{1 - \delta h^4((u^{\mathrm{T}} M_K u / u^{\mathrm{T}} M_L u) h^{-4})} = \frac{\lambda}{1 - \delta h^4 \mu}, \tag{24}$$

where $\lambda = u^{\mathrm{T}} K u / u^{\mathrm{T}} M_L u$ is the lowest eigenvalue computed with the lumped mass matrix M_L, and where, ultimately

$$\mu = \lim_{h \to 0} \frac{u^{\mathrm{T}} M_K u}{u^{\mathrm{T}} M_L u} h^{-4}.$$

We ascertain numerically that

$$\lambda = \pi^2 - 10.7 h^4 \quad \text{and} \quad \mu = 3.25,$$

so that

$$R[u] = \pi^2 + \frac{-10.7 + \pi^2 3.25 \delta}{1 - 3.25 h^4 \delta} h^4 + \text{higher-order terms}. \tag{25}$$

If $\delta = \frac{1}{3}$, or $\gamma = \frac{2}{3}$, then the $O(h^4)$ drops from $R[u]$ and we are left with $R[u] = \pi^2 + O(h^6)$.

Interestingly enough, the same $\gamma = \frac{2}{3}$ that assures the $O(h^6)$ accuracy in the finite element computation with the modified mass matrix assures the same accuracy from the Rayleigh quotient correction using the eigenvector computed from the lumped finite element mass matrix formulation. Fig. 1 depicts the convergence process of λ for the finite element consistent, lumped and optimal mass matrix formulations as well as for the Rayleigh quotient correction.

5. Linear triangular membrane element

From one dimension we pass to two and consider the membrane eigenproblem

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} + \lambda u = 0 \text{ in domain } D \text{ with } u = 0 \text{ on boundary } S. \tag{26}$$

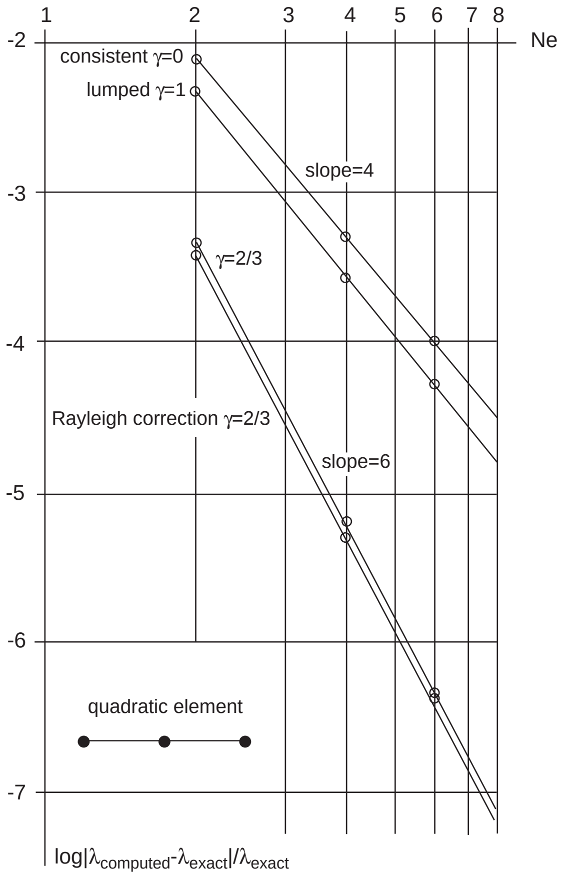

Fig. 1. Convergence of the fundamental eigenvalue λ of a unit string approximated by quadratic finite elements.

The finite element membrane matrices for a triangle of sides L_1, L_2, L_3 and area A are

$$k = \frac{1}{8A}\left(L_1^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} + L_2^2 \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} + L_3^2 \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} \right) \tag{27}$$

and

$$m(\gamma) = m_1 + \gamma m_2 = \frac{A}{12}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \gamma \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \right), \tag{28}$$

so that $m(\gamma = 0)$ is the consistent element mass matrix and $m(\gamma = 1)$ is the lumped.

We observe that

$$u^{\mathrm{T}}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}u = (u_1-u_2)^2+(u_2-u_3)^2+(u_3-u_1)^2 \tag{29}$$

for any vector u.

If $L_1 = L_2 = L_3 = h$, then the area of the equilateral triangle becomes $A = \sqrt{3}h^2/4$. Writing $\gamma = 1-\delta$ the element stiffness and mass matrices become

$$k = \frac{h^2}{8A}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and } m = \frac{A}{3}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{A\delta}{12}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \tag{30}$$

respectively. In view of Eq. (30) the element mass matrix may be written as

$$m = \frac{A}{3}I - \delta\frac{2}{3}\frac{A^2}{h^2}k, \tag{31}$$

where k is the element stiffness matrix. We use this element to discretize an equilateral triangular membrane of unit sides that is known [4] to have a fundamental eigenvalue of $\lambda = 16\pi^2/3 = 52.63789$. According to Ref. [1] finite element superconvergence is achieved here with $\gamma = \delta = \frac{1}{2}$.

To explore the correction available with the Rayleigh quotient we write it as

$$R[u] = \frac{u^{\mathrm{T}}Ku/u^{\mathrm{T}}M_Lu}{1-(\delta h^2/8)(u^{\mathrm{T}}Ku/u^{\mathrm{T}}M_Lu)} = \frac{\lambda}{1-(\delta h^2/8)\lambda} \tag{32}$$

for the fundamental eigenvector u computed from the finite element lumped matrix formulation. We numerically ascertain that $\lambda = 52.63789 - 170.4h^2$ so that

$$R[u] = 52.63789\frac{1-3.24h^2}{1-6.58\delta h^2} \tag{33}$$

with an optimum reached for the same $\delta = \gamma = \frac{1}{2}$.

Fig. 2 shows the convergence of λ for the equilateral fixed membrane of unit sides as the number of elements per side Nes is increased. Computation is shown for $\gamma = 0, 1, \frac{1}{2}$, and the Rayleigh correction. And indeed the accuracy of the computed λ is of $O(\mathrm{Nes}^{-2})$ for both $\gamma = 0$ and 1, but jumps to $O(\mathrm{Nes}^{-4})$ with $\gamma = \frac{1}{2}$ and the Rayleigh correction.

6. Square membrane

The next membrane element we consider is the four-nodes square of side h, with node 4 being opposite node 1, and node 3 being opposite node 2. Its element stiffness and mass matrices are

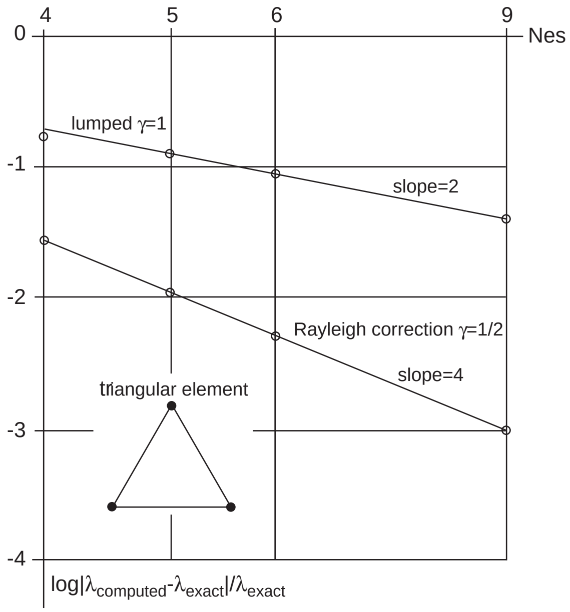

Fig. 2. Convergence of fundamental eigenvalue λ of a unit triangular membrane approximated by first-order elements.

$$k=\frac{1}{6}\begin{bmatrix}4&-1&-1&-2\\-1&4&-2&-1\\-1&-2&4&-1\\-2&-1&-1&4\end{bmatrix},\quad m(\gamma)=\frac{h^2}{36}\left(\begin{bmatrix}4&2&2&1\\2&4&1&2\\2&1&4&2\\1&2&2&4\end{bmatrix}+\gamma\begin{bmatrix}5&-2&-2&-1\\-2&5&-1&-2\\-2&-1&5&-2\\-1&-2&-2&5\end{bmatrix}\right), \quad (34)$$

respectively, with $m(\gamma=0)$ being the consistent element mass matrix and $m(\gamma=1)$ being the element lumped mass matrix.

We notice that

$$u^{\mathrm{T}}\begin{bmatrix}5&-2&-2&-1\\-2&5&-1&-2\\-2&-1&5&-2\\-1&-2&-2&5\end{bmatrix}u = \begin{array}{l}2(u_2-u_1)^2+2(u_4-u_3)^2+2(u_3-u_1)^2\\+2(u_4-u_2)^2+(u_3-u_2)^2+(u_4-u_1)^2\end{array} \quad (35)$$

for any vector u.

We use this element to discretize a unit square membrane for which $\lambda=2\pi^2$. Setting the fundamental eigenvector u computed with the lumped mass matrix into Rayleigh's quotient we obtain that

$$R[u] = 2\pi^2\left[\frac{1-2.47h^2}{1-3.28\delta h^2} + O(h^4)\right] \tag{36}$$

and at best $\delta = \frac{3}{4}$ or $\gamma = 1-\delta = \frac{1}{4}$, as for the optimal mass matrix.

7. Rectangular membrane

The element stiffness and mass matrices for the rectangular four-nodes membrane finite element of sides a and b are

$$k = \frac{b}{6a}\begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{bmatrix} + \frac{a}{6b}\begin{bmatrix} 2 & 1 & -2 & -1 \\ 1 & 2 & -1 & -2 \\ -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{bmatrix} \tag{37}$$

and

$$m(\gamma) = \frac{ab}{36}\left(\begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} + \gamma\begin{bmatrix} 5 & -2 & -2 & -1 \\ -2 & 5 & -1 & -2 \\ -2 & -1 & 5 & -2 \\ -1 & -2 & -2 & 5 \end{bmatrix}\right), \tag{38}$$

respectively.

To observe the influence of the membrane elongation on the optimal γ we propose to compute the fundamental frequency ω of a rectangular membrane of sides 1 and 2 fixed at its rim. The fundamental frequency of the membrane is

$$\omega^2 = \lambda = \pi^2\left(\frac{1}{1^2} + \frac{1}{2^2}\right) = \frac{5}{4}\pi^2. \tag{39}$$

We discretize the membrane by Nes elements per side, each being thus by itself of the aspect ratio $\frac{2}{1}$.

The γ computed for this membrane, both for the optimal mass matrix and the best Rayleigh quotient, turns out to be the same $\gamma = \frac{1}{4}$ as for the square membrane.

Fig. 3 shows the error in the computed first eigenvalue of the rectangular fixed membrane versus the number of elements per side Nes on a logarithmic scale. An identical graph is generated for the square membrane discretized by square elements.

8. Circular membrane

The circular membrane provides us with the simplest eigenproblem with variable coefficients. The element stiffness and mass matrices of a linear element located between radii r_1 and r_2 are

$$k = \frac{r_1 + r_2}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{40}$$

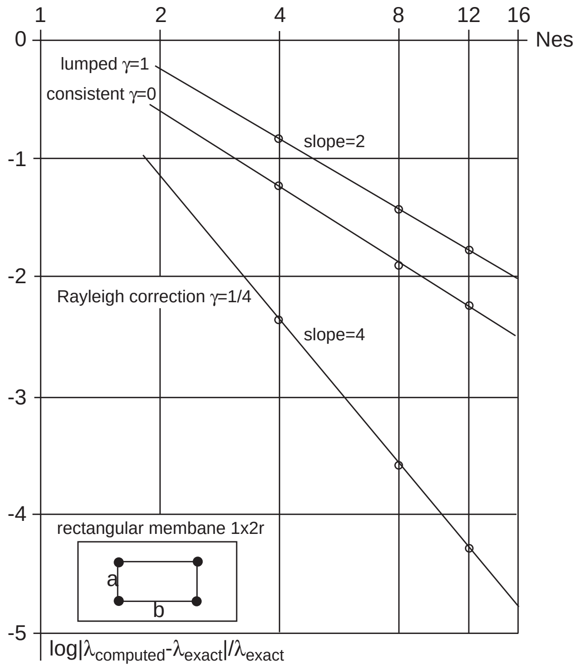

Fig. 3. Convergence of fundamental eigenvalue λ of a rectangular membrane.

and

$$m(\delta) = \frac{h}{4}(r_1 + r_2)\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \frac{h}{12}\delta\left(r_1\begin{bmatrix} & -1 \\ -1 & 2 \end{bmatrix} + r_2\begin{bmatrix} 2 & -1 \\ -1 & \end{bmatrix}\right), \tag{41}$$

respectively, where $\delta = 1 - \gamma$.

The lowest eigenvalue λ of a circular membrane of unit radius is obtained as the first root of Bessel's function, $J_0(\sqrt{\lambda}) = 0$, and $\lambda = 5.783186$. By the numerical analysis method of the previous sections we fix here an optimal $\delta = 0.95$ or $\gamma = 1 - \delta = 0.05$. Fig. 4 shows the dependence of the relative error in the computed λ upon γ around the optimal value.

Fig. 5 shows the convergence of the fundamental eigenvalue λ with the number of elements Ne in the discretization.

9. Higher modes

We return to Eq. (1) and the string problem discretized by the two-point linear finite element. The difference equation plus the end conditions $u_0 = u_{n-1} = 0$ is solved by

$$u_j = \sin \pi khj, \quad j = 0, 1, 2, \ldots, n+1, \quad k = 1, 2, \ldots, \tag{42}$$

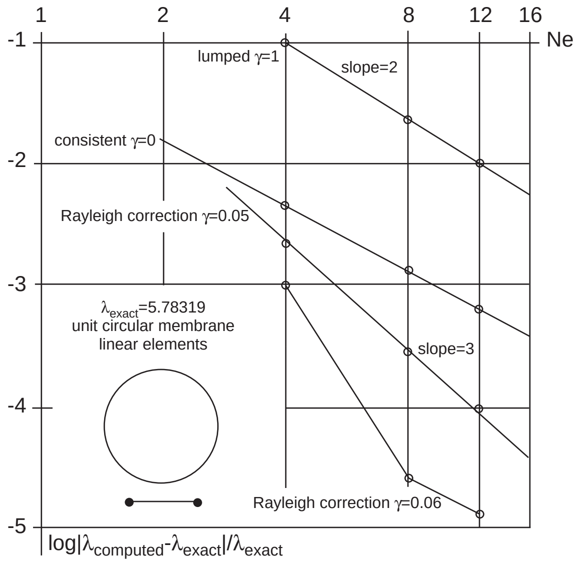

Fig. 4. Accuracy of the computed fundamental eigenvalue of a unit circular membrane versus γ.

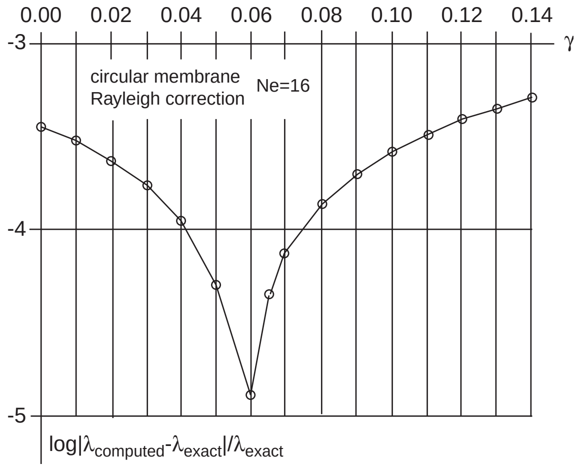

Fig. 5. Convergence of λ of a unit circular membrane with the number of elements, for different γ values.

where $h = 1/(n+1)$, with which we have

$$\lambda_k = \frac{12}{h^2} \frac{1 - \cos \pi kh}{4 + 2\gamma + 2(1-\gamma)\cos \pi kh} \tag{43}$$

as the approximation to the kth eigenvalue. We set $\delta = 1 - \gamma,\ x = \pi kh$, and have by power series expansion

$$\lambda_k = \pi^2 k^2 (1 + \tfrac{1}{12}(2\delta - 1)x^2 + \tfrac{1}{360}(1 - 10\delta + 10\delta^2)x^4 + \cdots) \tag{44}$$

in which, we recall, π^2k^2 is the exact kth eigenvalue of the unit string. Substituting $\delta = 0, 1$ into Eq. (44) we obtain

$$\lambda_k = \pi^2k^2(1 - \tfrac{1}{12}\pi^2k^2h^2 + \cdots), \quad \lambda_k = \pi^2k^2(1 + \tfrac{1}{12}\pi^2k^2h^2 + \cdots), \tag{45}$$

respectively, while for $\delta = \frac{1}{2}$ we have

$$\lambda_k = \pi^2k^2(1 - \tfrac{1}{240}\pi^4k^4h^4 + \cdots). \tag{46}$$

The error in λ_k computed with $\delta = 0$ and 1 is still $O(h^2)$ but is now proportional to k^2, that is, to the square of the mode index k. The error in λ_k computed with the optimal $\delta = \frac{1}{2}$ is still $O(h^4)$ but is now proportional to k^4, that is, to the fourth power of the mode index k.

As the mode index k rises the kth eigenfunction $u_k(x) = \sin k\pi x$ of the unit string becomes "wavier" in the sense that now

$$u_k''(x) = -\pi^2k^2 \sin k\pi x = -\lambda_k \sin k\pi x = -\lambda_k u_k, \quad 0 \leqslant x \leqslant 1 \tag{47}$$

and the ability of the linear finite elements to approximate it drops, as detailed in Ref. [5], and hence the accuracy decline in the computed higher frequencies.

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